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LETTER

Analytical relationship among nominal hardness, reduced Young's modulus, the work of indentation, and strain hardening exponent

Dejun Ma · Chung Wo Ong

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In an instrumented indentation test, the reduced modulus is expressed as:

$$E_r = \frac{\sqrt{\pi}}{2\beta}\frac{S_u}{\sqrt{A(h_{cm})}} \qquad (1)$$

where $S_u = dP/dh|_{h=h_m}$ is the initial unloading stiffness at maximum load P_m; $A(h_{cm})$, h_m, and h_{cm} are the corresponding projected contact area, maximum indentation depth, and maximum contact depth; and β is a correction factor. E_r is related to the Young's modulus and Poisson's ratio of the indented material (E, ν) and those of the indenter (E_i, ν_i) by the equation $1/E_r = (1-\nu^2)/E + (1-\nu_i^2)/E_i$, from which an estimate of E is derived if E_r is first determined. Obviously Eq. 1 indicates that the accuracy of the measured value of E_r (or E) relies on the reliability of the methods used to derive S_u and $A(h_{cm})$ (or h_{cm}), but S_u may vary substantially according to the condition of a test. For example, at low load condition load–displacement data are scattered, such that the value of S_u derived would have great uncertainty. In addition, $A(h_{cm})$ (or h_{cm}) estimated by applying the well-known Oliver and Pharr method [1, 2] can have a large error when the indented material is soft and shows weak work hardening. Improvement is achieved by applying an energy-based method, obtained by combining dimensional theorem and finite element simulations as reported in our recent work [3, 4]. In this method, an approximate relationship between the ratio of a nominal hardness to reduced Young's modulus (H_n/E_r), and the ratio of the work done during unloading to that during loading denoted as total work afterwards (W_e/W_t) was founded, in which the nominal hardness defined by $H_n \equiv P_m/A(h_m)$ is essentially different from the real hardness [5] $H \equiv P_m/A(h_{cm})$ and can be determined accurately by fully utilizing the accuracy of the measured load and displacement data from an instrumented indentation system. Consequently, E_r (and thus E) can be determined merely from H_n, W_e, and W_t. This approach is referred to as the pure energy method [6], and has been shown to be very successful. However, in our previous approach the relationship between H_n/E_r and W_e/W_t was derived entirely based on numerical simulations, while the physical insight and the subsequent analytic formulation were not achieved yet. In this work, we derive an equation of H_n/E_r as a function of W_e/W_t and hardening exponent n based on a more physical point of view in order to consolidate the physical basis of the method.

In the model of indentation under consideration, a three-dimensional rigid conical indenter with a given half angle, θ, is driven to indent into a homogeneous elastic–plastic solids with a yield strength, σ_y, strain hardening exponent, n, and Young's modulus, E, along the normal direction. The interface between the indenter and the solids is assumed to be frictionless. The first step of analysis is to set up a relationship correlating h_{cm} and h_m, with $\sigma_y/E \sim 0$ and n as a parameter. For this purpose, we consider the following two extreme cases. The first extreme case is for $\sigma_y/E \sim 0$ and $n = 0$. The indented material can equivalently be regarded as being rigid and perfectly plastic. Under this situation, $h_{cm} \approx 1.3h_m$ for a broad range of $52.5° \le \theta \le 80°$ [7]. In the second extreme case for

D. Ma
Department of Mechanical Engineering, The Academy of Armored Forces Engineering, Beijing 100072, People's Republic of China
e-mail: dejunma@yahoo.com

C. W. Ong (✉)
Department of Applied Physics and Materials Research Center, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, People's Republic of China
e-mail: apacwong@inet.polyu.edu.hk

$\sigma_y/E \sim 0$ and $n = 1$, the indented material is indeed ideally elastic, and $h_{cm} = (2/\pi)h_m$ [1]. As such, linear interpolation is applied to achieve an equation of $h_{cm} = [1.3 + (2/\pi - 1.3)n]h_m$ for any intermediate case with $\sigma_y/E \sim 0$ and $0 \le n \le 1$.

A further approximate relationship between h_{cm} and h_m with n and W_e/W_t as parameters is proposed as follows. When $\sigma_y/E \sim \infty$, the indented material can be regarded as ideally elastic, irrespective of the value of n. Under this situation, W_e/W_t approaches 1 and $h_{cm} = (2/\pi)h_m$. On the other hand, for the case of $\sigma_y/E \sim 0$ and $0 \le n \le 0.5$, the ratio W_e/W_t is nearly zero and $h_{cm} = [1.3 + (2/\pi - 1.3)n]h_m$. The proposed relationship should be consistent with the above extreme cases in the range of $0 \le \sigma_y/E \le \infty$ (or $0 \le W_e/W_t \le 1$) and $0 \le n \le 0.5$. Again, based on linear interpolation, we revise the above h_{cm}–h_m relationship as

$$h_{cm} = \left\{1.3 + \left(\frac{2}{\pi} - 1.3\right)n + \left(\frac{2}{\pi} - 1.3\right)(1-n)\left(\frac{W_e}{W_t}\right)\right\}h_m \quad (2)$$

The second step of analysis is to establish a relationship correlating power law index, m, of an unloading curve with W_e/W_t and n. According to Oliver and Pharr [1, 2], the unloading curve can be described as

$$P = B(h - h_f)^m \quad (3)$$

where B is a coefficient and h_f is the depth of the residual impression. The initial unloading stiffness is then determined as

$$S_u = \left.\frac{dP}{dh}\right|_{h=h_m} = \frac{mB(h_m - h_f)^m}{(h_m - h_f)} = \frac{mP_m}{(h_m - h_f)} \quad (4)$$

and the work done in the unloading process is given by

$$W_e = \int_{h_f}^{h_m} B(h - h_f)^m dh = \frac{P_m(h_m - h_f)}{m+1} \quad (5)$$

During loading, the force applied on the indenter is proportional to square of indentation depth according to dimensional analysis, that is

$$P = Ch^2 \quad (6)$$

The total work done during the loading process is obtained by

$$W_t = \int_0^{h_m} Ch^2 dh = \frac{1}{3}P_m h_m \quad (7)$$

From Eqs. 5 and 7, $h_m - h_f$ can be written as

$$h_m - h_f = \frac{m+1}{3}\left(\frac{W_e}{W_t}\right)h_m \quad (8)$$

The initial unloading stiffness S_u in Eq. 4 can then be rewritten as

$$S_u = \left.\frac{dP}{dh}\right|_{h=h_m} = \frac{3m}{(m+1)}\left(\frac{W_t}{W_e}\right)\frac{P_m}{h_m}. \quad (9)$$

The index m should be a function of σ_y/E (or W_e/W_t) and n. To obtain its function form, we assume that the unloading process is elastic, and is equivalent to a case of an elastic contact made between a flat elastic semi-space with elastic properties of E and v and an imaginary rigid indenter with a solid of revolution obeying a power law function, $z \equiv f(r) = kr^\alpha$, in which z is the axis of revolution, r the distance from the indenter surface to the axis, and k and α are constants. The condition for this hypothetical elastic contact interaction to be equivalent to a real unloading process is to let the pressure distributions at the peak load of these two cases identical. Yu and Blanchard [8] assumed that the distribution of the contact pressure for the case of a material with $\sigma_y/E \to 0$ and $n = 0$ indented by a rigid cone is uniform. One can thereby assume that the contact point experiences a contact pressure, p_0. Making use of the typical results for the hypothetical model of elastic indentation on a semi-space material, the elastic normal displacement at an arbitrary contact point can be determined as

$$w(r) = \frac{4(1-v^2)p_0 a}{\pi E}\int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{r}{a}\right)^2 \sin^2\varphi}\, d\varphi \quad (10)$$

where $r \le a$, and a is the maximum contact radius of the conical indent. On the other hand, according to the elastic contact theory, for a rigid indenter with the geometry obeying the power law function $z \equiv f(r) = kr^\alpha = (ka^\alpha)(r/a)^\alpha = k_1(r/a)^\alpha$, the function $w(r)$ should be in the form of:

$$w(r) = \delta - k_1(r/a)^\alpha \quad (11)$$

where δ is a constant. Through assigning different values of r/a ($0 \le r/a \le 1$) to the right side of Eq. 10, the parameter of α in Eq. 11 can be determined to be 2.523 by using least square fitting. Further, according to Sneddon [9], in the hypothetic case, P applied on the indenter with a power law function geometry, i.e., $z \equiv f(r) = kr^\alpha$, is proportional to $h_e^{1+1/\alpha}$, where $h_e = h - h_f$ is the indentation depth of the indenter into an elastic half space. Thus, the unloading function of Eq. 3 can be rewritten as

$$P = B(h - h_f)^m = B(h - h_f)^{1+1/\alpha} \quad (12)$$

where $m = 1 + 1/\alpha$. Consider the following extreme cases. First, for a material with $\sigma_y/E \to 0$ and $n = 0$, the value of $m = 1 + 1/\alpha = 1+1/2.523 = 1.396$. Second, for a material with $\sigma_y/E \to 0$ and $n = 1$; or $\sigma_y/E \to \infty$ and $0 \le n \le 1$, the value of $m = 2$. Therefore, by applying linear interpolation with n as a parameter, under the conditions of $\sigma_y/E \to 0$ and $0 \leq n \leq 1$, one obtains $m = 1.396 + 0.604\,n$. For the conditions of $0 \leq \sigma_y/E \leq \infty$ (or $0 \leq W_e/W_t \leq 1$) and $0 \leq n \leq 0.5$, following the same line of though for obtaining Eq. 2 for h_{cm}, one obtains:

$$m = 1.396 + 0.604n + 0.604(1-n)\left(\frac{W_e}{W_t}\right) \tag{13}$$

By substituting Eq. 9 and the contact area $A(h_{cm}) = \pi(h_{cm}\tan\theta)^2$ into Eq. 1, one obtains

$$\frac{P_m}{h_m^2 E_r} = \frac{2\beta\tan\theta}{3}\left(\frac{W_e}{W_t}\right)\left(\frac{h_{cm}}{h_m}\right)\left(\frac{m+1}{m}\right) \tag{14}$$

It needs to point out that the factor β in Eqs. 1 and 14 is dependent on the indenter geometry and the mode of deformation. For a conical indenter, both the analyses for the cases of linear elastic deformation [10] and small elastic–plastic deformation [11] give $\beta = 1$. In addition, the analysis for the case of large elastic–plastic deformation gives $\beta = 1.06$ when $\theta = 70.3°$ [12]. The effect of large deformation on β is expected to decrease with increasing θ [2]. Assuming that $\beta = 1$ when $\theta \to 90°$, the value of β can be expressed as a linear function of θ,

$$\beta = 1.2741 - 3.045 \times 10^{-3}\theta \tag{15}$$

Combining Eqs. 2, 13, 14, and 15, and using the definition of nominal hardness $H_n \equiv P_m/A(h_m) = P_m/[\pi(h_m\tan\theta)^2]$, the ratio of H_n/E_r can finally be expressed as a function of W_e/W_t and n, that is

$$\frac{H_n}{E_r} = \frac{2(1.2741 - 3.0455 \times 10^{-3}\theta)}{3\pi\tan\theta}\left(\frac{W_e}{W_t}\right) \times \left\{1.3 + \left(\frac{2}{\pi} - 1.3\right)n + \left(\frac{2}{\pi} - 1.3\right)(1-n)\left(\frac{W_e}{W_t}\right)\right\} \times \left\{1 + \frac{1}{1.396 + 0.604n + 0.604(1-n)\left(\frac{W_e}{W_t}\right)}\right\} \tag{16}$$

We note that this formula is effective in the range of $0 \leq W_e/W_t \leq 1$, $0 \leq n \leq 0.5$ and $52.5° \leq \theta \leq 80°$.

Consider a special case as an example. A conical indenter with $\theta = 70.3°$ is used. It has the same depth dependence of cross-sectional area as that of a Berkovich indenter. The relationships between H_n/E_r and W_e/W_t for $n = 0$ and $n = 0.45$, respectively, are derived as shown in Fig. 1. It is believed for any value of n between 0 and 0.45, the corresponding relationship of H_n/E_r and W_e/W_t should lie in the band bounded by the above two curves, and can thus be approximately expressed as a single value function, i.e., $H_n/E_r = f(W_e/W_t)$. Consequently, E_r can be obtained from the equation of $E_r = H_n/f(W_e/W_t)$ by only measuring the nominal hardness and the work of indentation. Considering a function $H_n/E_r = f(W_e/W_t)$ evaluated at $n = 0.25$, the maximum error of the estimated value of E_r should be bounded by $\lambda^+ = 1 - (H_n/E_r)|_{n=0.25}/(H_n/E_r)|_{n=0}$ and $\lambda^- = 1 - (H_n/E_r)|_{n=0.25}/(H_n/E_r)|_{n=0.45}$, which are calculated and shown in Fig. 2. Their magnitudes decrease almost linearly with increasing W_e/W_t. In particular, the maximum possible error occurs at $W_e/W_t \to 0$, which is determined to be ±16%. Obviously, from the engineering point of view, such a level of accuracy in the measurement of Young's modulus can meet the requirement of most applications. Also shown in Fig. 1 are the numerical results obtained from finite element simulations, from which it is seen that Eq. 16 can successfully predict the relationships of H_n/E_r and W_e/W_t. For the cases of $\theta = 75°$ and $\theta = 82°$, similar trends as those observed in the case of $\theta = 70.3°$ are found (Fig. 1). We conclude that good agreement between the analytical solutions and the finite element simulations shows that the present analysis reveals the essential relationships among H_n/E_r, W_e/W_t, and n, and therefore provides a physical basis for the pure energy method used to determine the Young's modulus of a material.

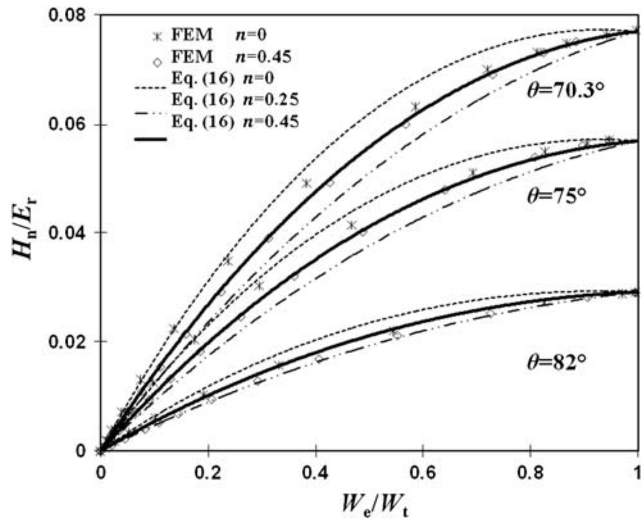

Fig. 1 Relationships among H_n/E_r, W_e/W_t, and n

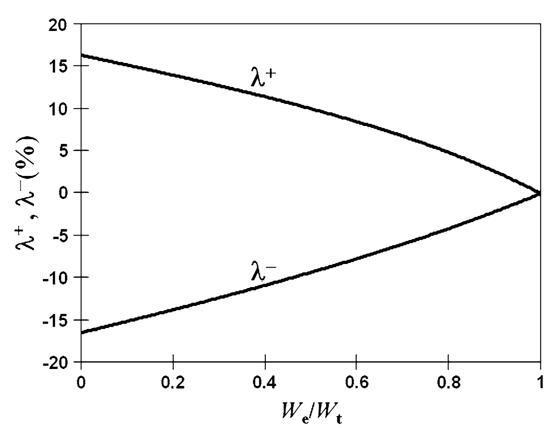

Fig. 2 Estimated maximum relative error λ^+ and λ^- versus W_e/W_t

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